

Lectures Delivered at Columbia University

ALGEBRAIC FUNCTIONS



Serge Lang

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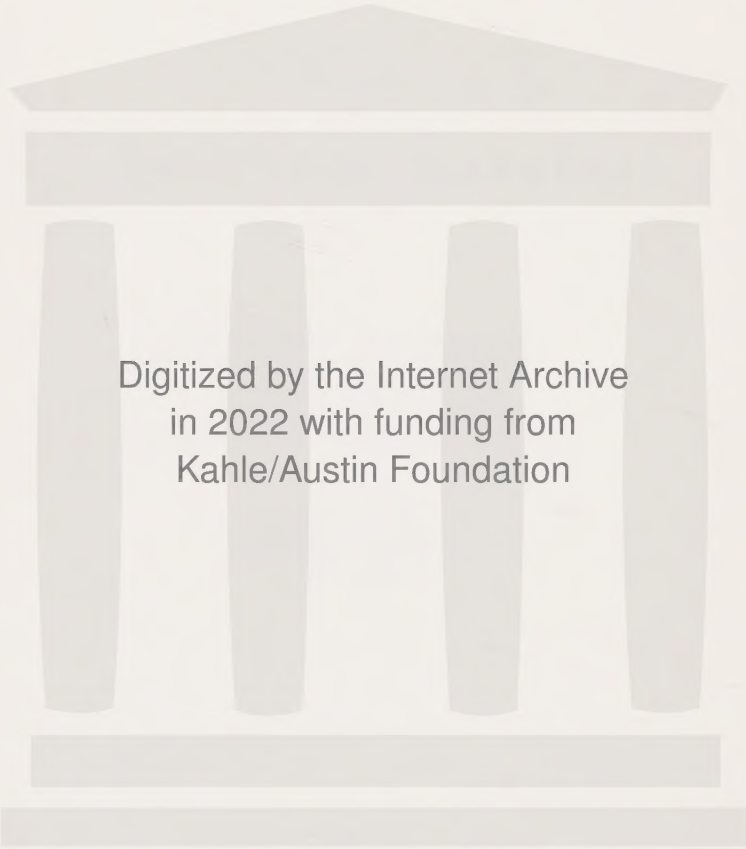
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Columbia University



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ALGEBRAIC FUNCTIONS

Introduction

The following exposes the basic facts concerning the Riemann surface of an algebraic function and a proof of the Abel-Jacobi theorem, as given by Artin in a seminar in 1948. The contents are as follows:

1. The Riemann surface as analytic manifold
2. Integration on R
3. Abelian integrals
4. Abel's theorem
5. Jacobi's theorem
6. Bilinear relations
7. Duality

We assume that the reader knows the Riemann-Roch theorem. (Cf. Chevalley's book on algebraic functions, or Introduction to Algebraic Geometry (IAG), Chapter X.) We shall also take for granted the existence of a representation of a compact orientable surface as a polygon with identified sides.

1. The Riemann Surface as Analytic Manifold

All fields will be assumed of characteristic 0.

Let K be a function field in one variable over a field k . This means that K is of transcendence degree 1, and finitely generated. If we can write $K = k(x,y)$, with two generators x, y , then we may view (x,y) as the generic point of a plane curve, defined by the equation $f(X,Y) = 0$, if f is the irreducible polynomial vanishing on (x,y) , determined up to a constant factor. A point (a,b) lies on the curve if and only if $f(a,b) = 0$. We shall say that the point is simple if $D_2 f(a,b) \neq 0$.

If $\mathfrak{o}$ is a discrete valuation ring of K over k (i.e., containing k) and $\mathfrak{m}$ its maximal ideal, then $\mathfrak{m}$ is principal, and any generator of $\mathfrak{m}$ is called a local parameter of $\mathfrak{o}$ or $\mathfrak{m}$. Assume that the residue class field $\mathfrak{o}/\mathfrak{m}$ is equal to k . Let $\varphi: \mathfrak{o} \rightarrow \mathfrak{o}/\mathfrak{m}$ be the canonical map. If $K = k(x,y)$ and $x, y \in \mathfrak{o}$, then we let $a = \varphi(x)$, $b = \varphi(y)$. We see that (a,b) is a point on the curve determined by (x,y) . If $z \in K$, $z \in \mathfrak{o}$, we can extend φ to all of K by letting $\varphi(z) = \infty$. We call φ a place of K (over k). We say that the point (a,b) is induced by the place on the curve.

LOCAL UNIFORMIZATION THEOREM. Let K be a function field in one variable over k .

1). Let $K = k(x,y)$ where (x,y) satisfy an irreducible polynomial $f(X,Y) = 0$ over k . Let $a, b \in k$ be such that $f(a,b) = 0$ but $D_2 f(a,b) \neq 0$. Then there exists a unique place φ of K over k such that $\varphi(x) = a$, $\varphi(y) = b$, and if $\mathfrak{o}$ is the corresponding discrete valuation ring with maximal ideal $\mathfrak{m}$, then $x-a$ is a generator of $\mathfrak{m}$.

2). Conversely, let $\mathfrak{o}$ be a discrete valuation ring of K containing k , with maximal ideal, $\mathfrak{m}$, such that $\mathfrak{o}/\mathfrak{m} = k$. Let x be a generator of $\mathfrak{m}$. Then there exists $y \in \mathfrak{o}$ such that $K = k(x,y)$, and such that the point induced by the place on the curve is simple.

Proof. To prove (1), we shall prove that any element $g(x,y) \in k[x,y]$ can be written in the form

$$g(x,y) = (x - a)^\nu \frac{A(x,y)}{B(x,y)}$$

where A, B are polynomials, and $B(a,b) \neq 0$. This proves that the ring $\mathfrak{o}$ consisting of all quotients of polynomials $g_1(x,y)/g_2(x,y)$ with $g_2(a,b) \neq 0$ is a discrete valuation ring, and that $x-a$ is a generator of its maximal ideal.

If $g(a,b) \neq 0$, we are done, so assume $g(a,b) = 0$. Write

$$\begin{aligned} g(a, Y) &= (Y - b)g_1(Y) & g_1(Y) &\in k[Y] \\ f(a, Y) &= (Y - b)f_1(Y) & f_1(Y) &\in k[Y] \end{aligned}$$

Then $f_1(b) \neq 0$ since $D_2 f(a,b) \neq 0$. Hence

$$g(a, Y)f_1(Y) = f(a, Y)g_1(Y)$$

It follows that

$$g(X, Y)f_1(Y) - f(X, Y)g_1(Y) = (X - a)A_1(X, Y)$$

for some polynomial A_1 . Hence

$$g(x,y) = (x - a)A_1(x,y)/f_1(y)$$

If $A_1(a,b) \neq 0$, we are done. If not, we continue in the same way. We cannot continue indefinitely, for otherwise, we know that there exists some place of K over k inducing the given point, and $g(x,y)$ would have a zero of infinite order at the discrete valuation ring belonging to that place, which is impossible.

Conversely, to prove (2), let $K = k(x,z)$ where z is integral

over $k[x]$. Let $z = z_1, \dots, z_n$ ($n \geq 2$) be the conjugates of z over $k(x)$, and extend $\mathfrak{o}$ to a valuation ring $\mathfrak{O}$ of $k(x, z_1, \dots, z_n)$. Let

$$z = a_0 + a_1 x + \dots + a_r x^r + \dots$$

be the power series expansion of z , with $a_i \in k$, and let $P_r(x) = a_0 + \dots + a_r x^r$. For $i = 1, \dots, n$ let

$$y_i = \frac{z_i - P_r(x)}{x^r}$$

If we take r large, then y_1 has no pole at $\mathfrak{O}$, but $y_2, \dots, y_n$ have poles at $\mathfrak{O}$. The elements $y_1, \dots, y_n$ are conjugate over $k(x)$. Let $f(X, Y)$ be the irreducible polynomial of (x, y) over k . Then

$$f(x, Y) = \psi_n(x) Y^n + \dots + \psi_0(x)$$

Furthermore, $\psi_i(0) \neq 0$ for some i , otherwise we could factor out some power of X from $f(X, Y)$. We rewrite $f(x, Y)$ in the form

$$f(x, Y) = \psi_n(x) y_2 \dots y_n (Y - y_1) \left(\frac{1}{y_2} Y - 1 \right) \dots \left(\frac{1}{y_n} Y - 1 \right)$$

In the valuation determined by $\mathfrak{O}$, we see that the coefficient

$$u = \psi_n(x) y_2 \dots y_n$$

cannot have a pole (otherwise divide the two expressions for $f(x, Y)$ by this coefficient and read the polynomial modulo the maximal ideal $\mathfrak{m}$ of $\mathfrak{o}$ to get a contradiction). If we denote by a bar the residue class of an element of $\mathfrak{o}$ mod $\mathfrak{m}$, then

$$0 \neq f(\bar{x}, Y) = (-1)^{n-1} \bar{u} (Y - \bar{y}_1)$$

By definition, $\bar{x} = a = 0$. We let $y = y_1$ and $\bar{y} = b$. Then

$$D_2 f(a, b) = (-1)^{n-1} \bar{u} \neq 0,$$

as was to be shown. *

COROLLARY. Let K be finite over $k(x)$. There is only a finite number of valuations of $k(x)$ over k which are ramified in K .

Proof. There exist only a finite number of points (a, b) such that $f(a, b) = 0$ and $D_2 f(a, b) = 0$, and there exist only a finite number of valuation rings of K such that x does not lie in the valuation ring (i.e., such that x is at infinity).

We let k be a locally compact field (so in characteristic 0 this means that k is the real field, complex field, or a p -adic field). By a prime or point of K we shall mean a discrete valuation ring of K over k , whose residue class field is k itself. (This should be called a prime rational over k .) A prime will be denoted by letters $P, Q, \dots$ and we think of primes as points. The local uniformization theorem shows in fact how to interpret primes as non-singular points on plane curves. We let R be the set of all primes of K over k , and call it the Riemann surface of K over k . We can view the elements of K as functions on R . If $P \in R$, we denote by $\mathfrak{o}_P$ the valuation ring associated with P , and by $\mathfrak{m}_P$ its maximal ideal. If $z \in \mathfrak{o}_P$, then we define $z(P)$ to be the residue class of z mod $\mathfrak{m}_P$. Thus $z(P) \in k$. If $z \notin \mathfrak{o}_P$, then we define $z(P) = \infty$. The elements of k are constant functions, and are in fact the only constant functions. If $z(P) = 0$, we say that z has a zero at P , and if $z(P) = \infty$, we say that z has a pole at P .

For each $x \in K$, we let $\Gamma_x = \{k, \infty\}$ be the Gauss sphere over k , i.e., the compact space obtained by adjoining to k a point at infinity. We let

$$\Gamma = \prod_{x \in K} \Gamma_x$$

We embed R in Γ in the obvious way: An element P goes on the product $\prod x(P)$. We topologize R as a subspace of Γ , which amounts to saying that the topology is the one having the least amount of open sets such that all the functions $x \in K$ are continuous.

The product Γ is compact, and we contend that R is closed in Γ . This implies that R is compact. Let $(a_x)_x \in K$ be in the closure. Let $\mathfrak{o}$ be the set of elements x in K such that $a_x \neq \infty$. Then $\mathfrak{o}$ is a valuation ring, whose corresponding point P is such that $x(P) = a_x$ for all x . This is easily proved. We first note that $k \subset \mathfrak{o}$ because $a(P) = a$ for all $a \in k$ and $P \in R$. Let $x, y \in \mathfrak{o}$. Then $a_x, a_y \neq \infty$. By assumption, there exists $P \in R$ such that $x(P)$, $y(P)$, and $(x + y)(P)$ are arbitrarily close to a_x , a_y , a_{x+y} respectively. For such P , we see that $x(P)$ and $y(P) \neq \infty$, whence $(x + y)(P) \neq \infty$. Hence $x + y \in \mathfrak{o}$. Similarly, $x - y$ and xy lie in $\mathfrak{o}$, which is therefore a ring. Furthermore, the map $x \mapsto a_x$ is a homomorphism of $\mathfrak{o}$ into k , and is the identity on k . This follows from a continuity argument as above. Finally, $\mathfrak{o}$ is a valuation ring, for suppose $x \in \mathfrak{o}$. Then $a_x \neq \infty$. Let $y = x^{-1}$. There exists $P \in R$ such that $x(P)$ is close to a_x and $y(P)$ is close to a_y . Since $x(P)$ is close to infinity, it follows that $y(P)$ is close to 0. Hence $a_y = 0$, so $y \in \mathfrak{o}$. This proves our assertion.

If P is a point, a generator t of the maximal ideal $\mathfrak{m}_P$ will be called a local uniformizing parameter at P . We shall now prove that the map

$$Q \mapsto t(Q)$$

gives a topological isomorphism of a neighborhood of P onto a neighborhood of 0 in k .

According to the local uniformizing theorem, we can find generators t, y for K such that the point P is represented by a simple point with coordinates (a, b) in k , and in fact

$a = 0$. Split the polynomial $f(0, Y)$ in the algebraic closure $\bar{k}$ of k . Then b is a root of multiplicity 1, so we have

$$f(0, Y) = (Y - b)(Y - b_2)^{e_2} \dots (Y - e_r)^{e_r}$$

The roots of a polynomial are continuous functions of the coefficients. There exists a neighborhood U of 0 in k such that for any element $\bar{t} \in U$, the polynomial $f(\bar{t}, Y)$ has exactly one root in $\bar{k}$, with multiplicity 1, and this root is close to b (in the algebraic closure of k). However, using, for instance, the Newton approximation method, starting with the approximate root b , we can refine b to a root of $f(\bar{t}, Y)$ in k itself, if we took U sufficiently small. Hence the map $Q \mapsto t(Q)$ is injective on the set of Q such that $(t(Q), y(Q))$ lies in a suitably small neighborhood of $(0, b)$. Since the topology on R is determined by the functions in K , and since k is locally compact, we conclude that for U sufficiently small, the map $Q \mapsto t(Q)$ gives a topological isomorphism of U onto a neighborhood of 0 in k .

We can choose U such that $t(U)$ is a disc on the t -plane. We observe that if Q is close to P , then applying the local uniformization theorem shows that $t - t(Q)$ is a local uniformizing parameter at Q , because the condition concerning the second partial derivative is satisfied by continuity. Indeed, if $D_2 f(a, b) \neq 0$, then $D_2 f(a', b') \neq 0$ for all (a', b') sufficiently close to (a, b) .

THEOREM 1. Taking as charts discs on t -planes as above, together with the maps given by local parameters, gives an analytic manifold structure to R .

Proof. If $P \in R$, and t, u are two parameters at P , then t has a power series expansion in terms of u , say

$$t = \sum_{i=0}^{\infty} a_i u^i$$

and since t is algebraic over $k(u)$, simple estimates show

that this power series is convergent in some neighborhood of the origin. Hence

$$t(Q) = \sum a_i u(Q)^i$$

for Q close to 0 in the t -plane, and we see that the functions giving changes of charts are holomorphic.

Theorem 1 is valid for any field k which is real, complex, or p -adic. From now on, we shall assume that $k = \mathbf{C}$ is the field of complex numbers.

We have the notion of a meromorphic function on R , i.e., a quotient of holomorphic functions locally at each point. If f is such a function, we can write it locally around a point P as a power series

$$f(Q) = \sum b_i t(Q)^i$$

where t is a parameter at P , and a finite number of negative powers of t may occur. In this way, f may be viewed as embedded in the power series field $K_P \approx \mathbf{C}((t))$, and thus can be viewed as an adele because it has only a finite number of poles on R since R is compact.

THEOREM 2. Every meromorphic function on R is in K .

Proof. Let L be the field of meromorphic functions on R . If $L \neq K$, then the degree $(L:K)_{\mathbf{C}}$ of the factor space of L mod K over the complex is infinite. But for the unit parallelopete, Λ_1 , consisting of adeles which are integral at all P , we have

$$(L:K)_{\mathbf{C}} = (L + \Lambda_1 : K + \Lambda_1)_{\mathbf{C}} + (L \cap \Lambda_1 + K \cap \Lambda_1)_{\mathbf{C}}$$

(cf. IAG, p. 236 ff.). The first term on the right is finite as it is shown in Weil's proof of the Riemann-Roch theorem. The second is 0 because a function having no pole is a constant (by the maximum modulus principle). Contradiction.

THEOREM 3. The Riemann surface is connected.

Proof. Let S be a connected component. Let $P \in S$ and let $z \in K$ be a function having a pole only at P (such a z exists by the Riemann-Roch theorem). Then z is holomorphic on any other component, without pole, hence constant, equal to c on such a component. But $z - c$ has infinitely many zeros, which is impossible since R and S are compact.

THEOREM 4. Let $z \in K$ be a non-constant function. The primes of K induce primes of $\mathbf{C}(z)$, and thereby induce a mapping of R onto of the z -sphere S_z , which is a ramified topological covering. The algebraic ramification index e_p at a point P of R is the same as the topological index, and the number of sheets of the covering is

$$n = [K : \mathbf{C}(z)].$$

Proof. Let P be a point of R and t a parameter at P . Then P induces a point $z = a$ on the z -sphere, and t^e is equal to $z - a$ times a unit in the power series ring $\mathbf{C}[[t]]$. Since one can extract n -th roots in $\mathbf{C}$, we can find a local parameter u at P on R such that $u^e = z - a$, where $a = z(P)$. The map

$$u(Q) \rightarrow u(Q)^e = z(Q) - a$$

gives an e to 1 map of the disc around P onto a disc V_a in the z -plane. We shall call such a disc regular for Q . We have shown that the topological ramification index is equal to e . As to the number of sheets, all but a finite number of primes of $\mathbf{C}(z)$ are unramified in K and, therefore, split completely into n primes of K (by the formula $\sum e_i = n$). Hence n is the number of sheets.

THEOREM 5. The Riemann surface is triangulable and orientable.

Proof. We shall triangulate it in a special way, used later

in another theorem. Let z be a non-constant function in K . Consider a triangulation of the z -sphere S_z such that every point q of S_z ramified in K is a vertex (just add such points to a given triangulation). A sufficiently high subdivision of the triangulation achieves the following properties.

If Δ is a triangle in S_z none of whose vertices is ramified, then Δ is contained in some regular neighborhood of a point, with ramification index equal to 1.

If Δ is a triangle with a ramified vertex Q , then $\Delta \subset V_Q$.

We can now lift the triangulation. First, each vertex lifts to a certain number of vertices on R , and each Δ none of whose vertices is ramified lifts uniquely to n triangles in R .

If Q is a point of R ramified above q in S_z , we let m be its ramification index and t a parameter at Q on R . We get a map

$$V_Q \rightarrow V_q$$

by

$$t \mapsto t^m = z - a$$

if we choose t suitably. Each point $z - a = re^{i\theta}$ has m inverse images

$$t = r^{\frac{1}{m}} e^{\frac{i\theta}{m} + \frac{2\pi i\nu\theta}{m}} \quad \nu = 1, \dots, m$$

Hence each triangle Δ with a vertex at q lifts in m ways.

As to orientability, we can assume that two triangles of our triangulation, none of whose vertices are ramified and having an edge in common, are contained in some regular disc. Secondly, if Δ has a ramified vertex q , and Δ_1 has an edge in common with Δ , then Δ, Δ_1 are both contained in V_q .

Now we can orient the triangles on S_z such that an edge

receives opposite orientation from the two triangles adjacent to it. If $\tilde{\Delta}$ on R covers Δ , we give $\tilde{\Delta}$ and its edges the same orientation as Δ . In view of our strengthened conditions, we can lift the orientation, so R is orientable.

Let now V, E, T be the number of vertices, edges, triangles on the z -sphere. Then denoting by a prime the same objects on R , we get

$$E' = nE \quad T' = nT$$

On the other hand, let r be the number of primes p of $\mathbf{C}(z)$ which are ramified, or $p = \infty$. Let n_p be the number of primes P of K such that P lies above p . Then

$$V' = n(V - r) + \sum_p n_p$$

the sum being taken over ramified p or infinity. We have:

$$\begin{aligned} V' &= nV - nr + \sum_p n_p \\ &= nV + \sum_p (n_p - n) \\ &= nV - \sum_{P|p} \sum_p (e_P - 1) \\ &= nV - \sum_P (e_P - 1) \end{aligned}$$

where this last sum is taken over all primes P of K .

Let χ_Z be the Euler characteristic of S_Z and χ_R that of R . We shall prove:

THEOREM 6. Let g be the topological genus of R . Then

$$2g - 2 = -2n + \sum_P (e_P - 1)$$

Proof. We have $\chi_Z = B_0 - B_1 + B_2$ where B_i is the i -th

Betti number. We have $B_0 = B_2 = 1$. But also $\chi_Z = V - E + T$, so $\chi_Z = 2$.

Now $\chi_R = V' - E' + T'$ and by our previous result, this is

$$\begin{aligned} &= nV - \sum (e_P - 1) - nE + nT \\ &= n\chi_Z - \sum (e_P - 1) \end{aligned}$$

But $B'_0 = B'_2 = 1$ and $B'_1 = 2g$. Hence $\chi_R = 2 - 2g$, whence

$$2g - 2 = -2n + \sum (e_P - 1)$$

as desired.

COROLLARY. The algebraic genus is equal to the topological genus.

Proof. Both satisfy the same formula.

2. Integration on R

As before, K is a function field over $\mathbf{C}$ and R its Riemann surface. We shall follow Chevalley to define integration.

If U is an open subset of R , we can define the notion of meromorphic differential on U , namely an expression of type fdx where f is a meromorphic function on U and x lies in K . We say that $fdx \sim gdx$ if $f/g = dy/dx$. We say that the differential is holomorphic at P if, whenever t is a parameter at P , and

$$fdx = f \frac{dx}{dt} dt$$

at P , then the function fdx/dt has no pole at P . We say that the differential is holomorphic on U if it is so at every point of U .

A primitive g of fdx on U is a meromorphic function on U

such that $dg/dx = f$. If U is connected, then two primitives differ by a constant, as always.

If ω is a holomorphic differential on a disc V , then ω has a primitive g and g is holomorphic on V .

If γ is a 1-simplex contained in a disc V , and $\omega = f dx$ on V , and $\partial\gamma = P - Q$, then we define

$$\int_{\gamma} \omega = f(P) - f(Q)$$

This number is independent of the choice of V and f . Indeed, if $\text{supp } (\gamma)$ is contained in another disc V_1 , and g_1 is a primitive of fdx on V_1 , then $V \cap V_1$ contains a connected open set W containing γ and $g - g'$ is constant on W .

If $\text{subd}_1 \gamma = \gamma_1 + \gamma_2$ then $\partial\gamma_1 = P - Q$, $\partial\gamma_2 = Q - A$ and

$$\int_{\gamma} \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega$$

Hence if γ is contained in a disc V , then

$$\int_{\gamma} \omega = \int_{\text{subd}_1 \gamma} \omega$$

If $\gamma = \sum n_i \sigma_i$ where σ_i is a 1-simplex contained in some open disc we define the integral of ω over γ by linearity. We get again that the integral does not change by a subdivision.

If γ is any 1-chain, for r large enough, each member of the r -th subdivision is contained in some open disc, and we can define the integral over a 1-chain, independently of the subdivision chosen subject to this condition.

CAUCHY'S THEOREM. Let ω be holomorphic on an open set U of $\mathbb{R}$. Let γ be a ℓ -cycle on U , homologous to 0 on U . Then

$$\int_{\gamma} \omega = 0$$

Proof. $\gamma = \partial\eta$ where η is a 2-chain. For some r , $Sd^r\gamma$ has each one of its simplices contained in a disc. We have

$$Sd^r\gamma = Sd^r\partial\eta = \sigma Sd^r\eta$$

whence it suffices to prove Cauchy's theorem under the assumption that η and γ are contained in a disc. But then ω has a primitive, and the result is trivial.

COROLLARY. If U is simply connected, then ω has a primitive on U .

3. Abelian Integrals

A differential will be said to be of the first kind if it is holomorphic everywhere on the Riemann surface. Such differentials form a vector space over the complex, and by the Riemann-Roch theorem, one sees that the dimension of this space is equal to the genus g of R .

From topology, we now take for granted that our Riemann surface can be represented as a polygon $\mathcal{P}$ with identified sides (Fig. 1).

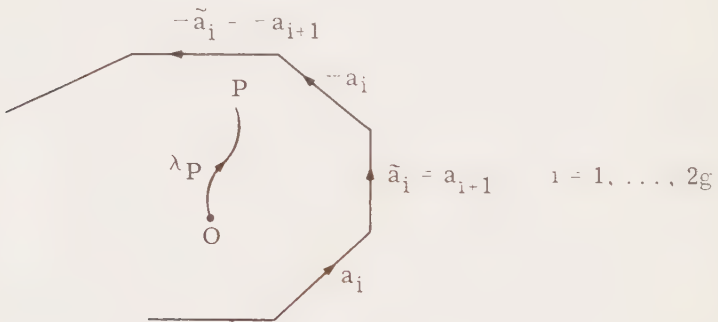


Figure 1

We select a point O inside the polygon, and use it as an origin. Let φ be a differential of first kind (written dfk). Given any

point P inside the polygon (which is simply connected), we define

$$f(P) = \int_{\lambda_P} \varphi$$

where λ_P is any path from O to P lying entirely inside the polygon. Then f is single valued and holomorphic inside $\mathcal{O}$.

If γ_1 is an arbitrary path on R , not necessarily lying inside the polygon, from O to P , then

$$\lambda_1 \sim \lambda_P + \sum n_i a_i$$

with suitable integers (and $\sim$ means homologous). Consequently

$$\int_{\lambda_1} \varphi = \int_{\lambda_P} \varphi + \sum n_i \int_{a_i} \varphi$$

The numbers

$$a_i = \int_{a_i} \varphi$$

generate an abelian group which will be called the group of periods of φ . An integral taken from O to P along any path is well defined modulo periods.

The integral over the path λ as shown in Fig. 2

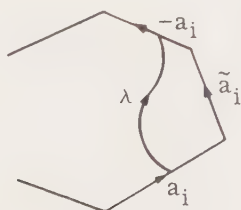


Figure 2

is also a period, and we write

$$\int_{\lambda} \varphi = \tilde{a}_i$$

Consider one of the cycles a_i . We can find a simply connected open set U containing a_i minus the vertex v . We can now define a holomorphic function f_i^+ on U as follows. Given any point P on $a_i - v$, we take a path λ_P^+ lying entirely inside the polygon except for its end point. For any point Q in U we then define

$$f_i^+(Q) = \int_{\lambda_P^+} \varphi + \int_P^Q \varphi$$

the integral from P to Q being taken on a path lying entirely inside U . Our function f_i^+ is holomorphic on $a_i - v$.

Let ω be a differential which is holomorphic on a_i (including the vertex). We can define

$$\int_{a_i} f_i^+ \omega$$

as the limit of the integral

$$\int_{P_i}^{Q_i} f_i^+ \omega$$

as P_i, Q_i approach the vertex as indicated.

Similarly, we define f_i^- as the similar object on the side $-a_i$. We get:

$$\int_{-a_i} f_i^- \omega = - \int_{a_i} f_i^- \omega$$

We also observe that $f_i^+(P) - f_i^-(P) = -\tilde{a}_i$ for P on a_i . We can now define the symbol $\int_{\mathcal{O}} f \omega$ for any differential holomorphic on the polygon $\mathcal{O}$ by

$$\begin{aligned}
\int_{\mathcal{O}} f\omega &= \sum \int_{a_i} f_i^+ \omega + \sum \int_{-a_i} f_i^- \omega \\
&- \sum \lim_{\substack{P_i \rightarrow v \\ Q_i \rightarrow v}} \int_{P_i}^{Q_i} (f_i^+ - f_i^-) \omega \\
&- \sum \lim_{P_i} \int_{P_i}^{Q_i} \tilde{a}_i \omega \\
&= -\sum \tilde{a}_i w_i
\end{aligned}$$

where $w_i = \int_{a_i} \omega$ is a period of ω .

THEOREM 7. Let ω be a differential holomorphic on $\mathcal{O}$.
Then $\int_{\mathcal{O}} f\omega$ is given by

$$-\sum w_i \tilde{a}_i = 2\pi \sqrt{-1} \sum \text{res}_P(f\omega)$$

where the sum is taken over poles of ω .

Proof. Our function f is uniquely defined inside the polygon. To get the integral over the polygon itself, where f has been defined separately for the two representations of the same side, we approximate the polygon $\mathcal{O}$ by a polygon $\mathcal{O}'$ as indicated (Fig. 3)

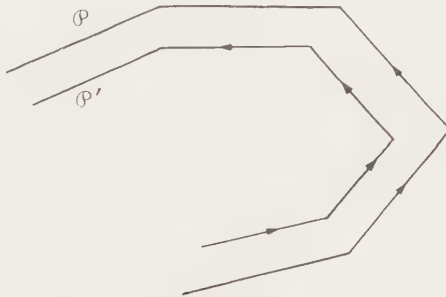


Figure 3

such that all the poles of ω lie inside $\mathcal{O}'$. Then

$$\mathcal{P}' \xrightarrow{\quad} \mathcal{P} \quad \int_{\mathcal{P}'} f\omega = \int_{\mathcal{P}} f\omega$$

the limit being taken as the vertices of $\mathcal{P}'$ approach the vertex of $\mathcal{P}$. The polygon $\mathcal{P}'$ is homologous to the sum of small circles taken with positive orientation around the poles of ω , and the integral around $\mathcal{P}'$ is, therefore, constant, equal to the expression on the right hand side of our equality. Since it approaches the integral around $\mathcal{P}$, it is equal to it, thereby proving our theorem.

We formulate our preceding results more generally as follows:

Let $\Phi = (\varphi_1, \dots, \varphi_g)$ be a basis for the dfk. We consider the vector integral

$$\int_{a_i} \Phi = \left(\int_{a_i} \varphi_1, \dots, \int_{a_i} \varphi_g \right) = (\alpha_{i1}, \dots, \alpha_{ig}) = A_i$$

and call it a vector period, or simply period, from now on. Similarly,

$$\int_{\tilde{a}_i} \Phi = (\tilde{\alpha}_{i1}, \dots, \tilde{\alpha}_{ig}) = \tilde{A}_i$$

The modules of periods generated over the integers by $A_1, \dots, A_{2g}$ and $\tilde{A}_1, \dots, \tilde{A}_{2g}$ are the same, and in fact these two sets differ by a permutation.

For P inside the polygon, we let

$$F(P) = \int_{\lambda_P} \Phi = (f_1(P), \dots, f_g(P))$$

the f_ν corresponding to integrals of φ_ν . We define F_i^+ and F_i^- in a similar way, so that for P on a_i (and $\neq$ vertex) we have

$$F_i^+(P) = F_i^-(P) = -\tilde{A}_i$$

We can now define the vector integral $\int_{\mathcal{O}} F\omega$ and Theorem 1 can be formulated vectorially.

THEOREM 8. If ω is holomorphic on $\mathcal{O}$, then $\int_{\mathcal{O}} F\omega$ is equal to $-\sum_i w_i \tilde{A}_i - 2\pi\sqrt{-1} \sum \text{res}_P(F\omega)$ the residue being the vector of residues, and the sum is taken over all poles of ω .

4. Abel's Theorem

We have fixed a point O on the Riemann surface. The vector integral

$$\int_O^P \Phi = \left(\int_O^P \varphi_1, \dots, \int_O^P \varphi_g \right)$$

as P varies over the Riemann surface, taken along some path from O to P is well defined modulo the periods. In this way, we obtain a mapping from the Riemann surface R into the factor space $\mathbb{C}^g$ modulo of periods. This mapping can be extended by linearity to the free abelian group generated by points, this group being called the group of cycles on R , or divisors. A divisor is a formal sum

$$a = \sum n_P P$$

with integers n_P , almost all of which are 0. Its degree is the sum of the n_P . Those of degree 0 form a group D_0 , and for these, it is clear that the restriction of the above homomorphism is then independent of the origin chosen. We thus obtain a homomorphism

$$a \mapsto S(a) \equiv \sum n_P \int_O^P \Phi$$

modulo periods from D_0 into $\mathbb{C}^g/(\text{periods})$. This factor group will be denoted by J and be called the Jacobian of R . The main theorem states:

ABEL-JACOBI THEOREM. The preceding homomorphism from D_0 into J is surjective, and its kernel consists of the divisors of functions. It, therefore, establishes an isomorphism between the divisor classes (for linear equivalence) of degree 0, and the Jacobian group $\mathbb{C}^g$ modulo periods.

The statement concerning the kernel of our homomorphism is called Abel's theorem and will be proved in this section. The surjectivity is postponed to the next section.

We first prove that divisors of functions are contained in the kernel. In other words, if z is a function with divisor

$$(z) = \sum n_P P$$

we have to show that

$$\sum n_P F(P) \equiv 0 \text{ mod periods}$$

We can always find a polygon representation of R such that z has no pole on $\mathcal{O}$, and we let $\omega = dz/z$ in Theorem 8. Then $\text{res}_P(F\omega) = n_P F(P)$. Indeed, for any holomorphic function f consider a local parameter t at a point P . Then $f\omega = f(dz/dt)z^{-1} dt$. Also,

$$z^{-1} \frac{dz}{dt} = \frac{k}{t} + \dots$$

if z has order k at P . Thus

$$\text{res}_t \left(f z^{-1} \frac{dz}{dt} \right) = f(0)k$$

By Theorem 8,

$$2\pi\sqrt{-1} \sum n_P F(P) = - \sum w_i \tilde{A}_i$$

where $w_i = \int_{a_i} z^{-1} dz$. But it is easily shown that $w_i = 2\pi\sqrt{-1}m_i$ for some integer m_i . Hence we cancel $2\pi\sqrt{-1}$ and we get

$$\sum n_P F(P) = -\sum m_i \tilde{A}_i$$

as was to be shown.

In order to prove the converse, we need some lemmas.

LEMMA 1. Let x_i ($i = 1, \dots, 2g$) be complex numbers. Then $\sum x_i \tilde{A}_i = 0$ if and only if $x_i = B \cdot A_i$ for some complex vector $B = (b_1, \dots, b_g)$. The vectors $A_1, \dots, A_{2g}$ span a g -dimensional space over the complex numbers.

Proof. Let $\psi = b_1\varphi_1 + \dots + b_g\varphi_g$ be a dfk. Then $F\psi$ has no poles and hence by Theorem 8,

$$0 = -\sum \text{res}_P(F\psi) = \frac{-1}{2\pi\sqrt{-1}} \sum w_i \tilde{A}_i$$

where

$$w_i = \sum_{\nu=1}^g b_\nu \int_{a_i} \varphi_\nu = \sum b_\nu a_{i\nu} = B \cdot A_i$$

This proves half the theorem.

To prove the converse, we prove first that the relation $X \cdot A_i = 0$ for all i implies that $X = 0$ (for a complex vector X). Let $\omega = X \cdot \Phi$. Then

$$\int_{a_i} \omega = \int_{a_i} X \cdot \Phi = X \cdot \int_{a_i} \Phi = X \cdot A_i = 0$$

by hypothesis. Hence all the periods of ω are 0. Hence the integral $\int_O^P \omega$ is a holomorphic function on R without poles because ω is of first kind. It is, therefore, constant

hence $\omega = 0$ and $X = 0$ because the φ_ν are linearly independent over $\mathbb{C}$.

We have now shown that the periods generate a vector space of dimension g over the complex. We recall that the A_i and the $\tilde{A}_i$ both generate the module of periods. To prove the last assertion of our theorem, we are trying to find all solutions of the system

$$(*) \quad \begin{cases} x_1 a_{11} + \cdots + x_{2g} a_{2g,1} = 0 \\ \quad \quad \quad \cdots \\ x_1 a_{1g} + \cdots + x_{2g} a_{2g,g} = 0 \end{cases}$$

The column rank is equal to the row rank, and this rank is exactly g by the above assertion. Thus the space of solutions has dimension g . On the other hand, the solutions $X = (B \cdot A_1, \dots, B \cdot A_{2g})$ form a g -dimensional space, which must, therefore, be the whole space of solutions. Our lemma is proved.

A differential ω is said to be of second kind if its residue is 0 at all points. It is said to be of third kind if it has at most poles of order 1 at all points. The next lemma concerns such differentials.

LEMMA 2. Let $\alpha = \sum n_P P$ be a divisor of degree 0. Then there exists a differential of third kind ω such that $\text{res}_P \omega = n_P$ for all P .

Proof. It suffices to prove our assertion for the case where all coefficients are zero except for two points. Indeed, by induction, suppose it proved for points $P_1, \dots, P_r$. Let Q be a new point unequal to the given points. Let $\omega_j (j = 1, \dots, r)$ have poles only at P_j and Q such that $\text{res}_{P_j} \omega_j = n_j$ and $\text{res}_Q \omega_j = -n_j$. We let $\omega = \sum \omega_j$. It obviously has the required property.

The Riemann-Roch theorem gives

$$\ell(-\alpha) = -\deg(\alpha) + 1 - g + d(\alpha)$$

where $d(\alpha)$ is the dimension of the space of differentials ω such that $(\omega) \geq -\alpha$. Put $\alpha = P_1 + P_2$. We get $d(\alpha) = g + 1$ and hence there exists a dtk ω having a pole of order 1 at P_1 or P_2 . But the sum of the residues is 0, so it actually has a pole at both, with opposite residues. Multiplying ω by a suitable constant gives what we want.

We can now determine the kernel of our map. Let $\alpha = \sum n_P P$ be a divisor of degree 0 such that $S(\alpha) = 0$. We have to show that α is the divisor of a function.

We contend that there exists a dtk ψ such that ψ has poles at P with residues n_P , and such that

$$\int_{a_i} \psi = 2\pi\sqrt{-1} n_i$$

for suitable integers n_i . By the lemma, there exists a dtk ω having residue n_P at P for all P . For any P , we have

$$\text{res}_P(F\omega) = n_P F(P)$$

By Theorem 8, we get

$$\begin{aligned} \sum w_i \tilde{A}_i &= 2\pi\sqrt{-1} \sum \text{res}_P(F\omega) \\ &= 2\pi\sqrt{-1} \sum n_P F(P) \\ &= 2\pi\sqrt{-1} \sum m_i \tilde{A}_i \quad (\text{by hypothesis}) \end{aligned}$$

Hence

$$\sum (w_i - 2\pi\sqrt{-1} m_i) \tilde{A}_i = 0$$

By lemma 1, we have

$$w_i - 2\pi\sqrt{-1} m_i = B \cdot \tilde{A}_i$$

for some vector B . We let $\psi = \omega - B \cdot \Phi$. Then ψ has the same poles and residues as ω . Furthermore,

$$\int_{a_i} \psi = \int_{a_i} \omega - \int_{a_i} B \cdot \Phi = w_i - B \cdot A_i = 2\pi\sqrt{-1} m_i$$

thereby proving our contention.

To construct the desired function z we take essentially

$$z(P) = \exp \int_O^P \psi$$

O being as usual a suitable origin inside our polygon. For any point P which is not a pole of ψ , and any path from O to P , the exponential of the integral gives a function which is independent of the path and is thus a well defined meromorphic function. For a pole of ψ , expanding ψ in terms of a local parameter around this point shows at once that we get a meromorphic function around the point, whose singularity can only be a pole. Abel's theorem is proved.

5. Jacobi's Theorem

We must now prove that our map from divisors of degree 0 into the factor group $\mathbf{C}^g$ modulo periods is surjective.

A divisor α is said to be non-special if $d(-\alpha) = 0$, i.e., if there exists no differential ω such that $(\omega) \geq \alpha$.

LEMMA 3. There exists g distinct points $M_1, \dots, M_g$ such that the divisor $M_1 + \dots + M_g$ is non-special.

Proof. Let $\omega_1 \neq 0$ be a dfk, and M_1 a point which is not zero of ω_1 . The space of dfk having a zero at M_1 has dimension $g - 1$ by Riemann-Roch. Let $\omega_2 \neq 0$ be in it and let M_2 be a point which is not a zero of ω_2 . Continue g times to get g points, as desired.

THEOREM 9. Let $M_1, \dots, M_g$ be g distinct points such that the divisor $M_1 + \dots + M_g$ is non-special. Then the map

$$(P_1, \dots, P_g) \rightarrow \sum \int_{M_i}^{P_i} \Phi$$

gives a homomorphism of a product of small discs

$V_1 \times \dots \times V_g$ around the points $P_1, \dots, P_g$ onto a neighborhood of zero in $\mathbb{C}^g$.

Before proving our theorem, we show how it implies Jacobi's theorem.

First we note that it suffices to prove Jacobi's theorem for a neighborhood of zero in $\mathbb{C}^g$, i.e., to prove local surjectivity.

Indeed, let X be any vector. For large n , $1/n \cdot X$ is in a small neighborhood of zero so, by the local result, we can find a divisor of degree zero such that $F(a) \equiv 1/n \cdot X$ modulo periods. Then $F(na) \equiv X$ modulo periods.

Using the notation of Theorem 9, let a range over divisors $P_1 + \dots + P_g - (M_1 + \dots + M_g)$, with P_i ranging over V_i . Then clearly

$$F(a) \equiv \sum_{M_i}^{P_i} \Phi \quad (\text{modulo periods})$$

and the theorem implies what we want.

We now prove Theorem 9. Let t_i be a local parameter at M_i . We contend that the determinant

$$\begin{vmatrix} \frac{\varphi_1}{dt_1}(M_1) & \dots & \frac{\varphi_1}{dt_g}(M_g) \\ & \dots & \\ \frac{\varphi_g}{dt_1}(M_1) & \dots & \frac{\varphi_g}{dt_g}(M_g) \end{vmatrix}$$

is not zero. To prove this, consider the homomorphism

$$\varphi \mapsto \left[\frac{\varphi}{dt_1}(M_1), \dots, \frac{\varphi}{dt_g}(M_g) \right]$$

of differentials of first kind into complex g -space. Any dfk in the kernel would be special, so the kernel is trivial, and our map is a linear isomorphism. Hence our determinant is non-zero.

We write

$$\begin{aligned}\varphi_1 &= h_{11}(t_1)dt_1, \dots, \varphi_1 = h_{1g}(t_g)dt_g \\ &\dots \\ \varphi_g &= h_{g1}(t_1)dt_1, \dots, \varphi_g = h_{gg}(t_g)dt_g\end{aligned}$$

where the $h_{ij}(t_j)$ is holomorphic in V_j . Let $h_{ij}^*(t_j)$ be the integral of the power series $h_{ij}(t_j)$, normalized to vanish at $t_j = 0$. Then we get a representation of our map:

$$\begin{aligned}(P_1, \dots, P_g) &\mapsto (h_{11}^*(t_1) + \dots + h_{1g}^*(t_g), \dots) \\ &= (H_1(t_1, \dots, t_g), \dots, H_g(t_1, \dots, t_g))\end{aligned}$$

where each $H_i(t_1, \dots, t_g)$ is holomorphic in the g variables, on $V_1 \times \dots \times V_g$. Now

$$\frac{\partial H_1}{\partial t_j} = h_{1j}(t_j)$$

and hence the Jacobian determinant evaluated at $(0, \dots, 0)$ is precisely our preceding determinant, and is non-zero. By the implicit function theorem, we get our local analytic isomorphism, thereby proving Jacobi's theorem.

The theorem can be complemented by an important remark. We consider complex g -space as a real $2g$ -dimensional space. We have:

THEOREM 10. The periods $A_1, \dots, A_{2g}$ are linearly independent over the reals. Hence the factor space of $\mathbb{C}^g$ by the periods is a $2g$ -real-dimensional torus.

Proof. It will suffice to prove that any complex vector X is congruent to a vector of bounded length modulo the periods. By the Riemann-Roch theorem, any divisor of degree 0 is

linearly equivalent to a divisor of type $(P_1 + \cdots + P_g) - g.O$, for suitable points $P_1, \dots, P_g$, which may be viewed as lying inside or on the polygon. For P ranging over the Riemann surface, the integrals

$$\int_O^P \Phi$$

are bounded in the Euclidean norm, if we take the path of integration to be entirely inside the polygon, except if P lies on the boundary. Consequently, the sum

$$\sum_{i=1}^g \int_O^{P_i} \Phi$$

has also bounded norm (g times the other bound). Combining this with Jacobi's surjectivity theorem shows that any vector is congruent to one with bounded length modulo periods, and concludes the proof of Theorem 10.

6. Bilinear Relations

Let φ, φ' be two dfk, and consider the polygon representation of R , with sides $a_i, b_i, -a_i, -b_i$ ($i = 1, \dots, g$) following each other (Fig. 4).

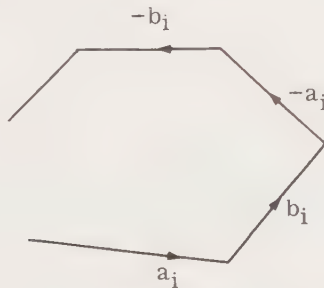


Figure 4

We let $\alpha_i = \int_{a_i} \varphi$ and $\beta_i = \int_{b_i} \varphi$ be the canonical periods of φ , and α'_i, β'_i those of φ' .

RIEMANN'S RELATIONS state that

$$\sum_1^g (\alpha_i \beta'_i - \alpha'_i \beta_i) = 0$$

Proof. Let $f(P)$ be the function defined before (integral from O to P , suitably defined on the boundary of the polygon), and consider the integral

$$\int_{\varphi} f \varphi'$$

Since φ' has no poles, we get from Theorem 8:

$$0 = \sum_{i=1}^g \left(\int_{a_i} \varphi' \int_{b_i} \varphi + \int_{b_i} \varphi' \int_{-a_i} \varphi \right)$$

which gives what we want.

7. Duality

LEMMA 4. Numbering the sides of the polygon as at the beginning, $a_1, \dots, a_{2g}$, and given an index i , there exists a dfk φ with periods satisfying

$$\operatorname{Re} \int_{a_i} \varphi = 1 \quad \operatorname{Re} \int_{a_j} \varphi = 0 \quad \text{for } j \neq i$$

Proof. Consider the period matrix:

	a_1	$\dots$	a_{2g}
φ_1	$u_1^1 + i v_1^1$		$u_{2g}^1 + i v_{2g}^1$
$\vdots$		$\dots$	
φ_{2g}	$u_1^{2g} + i v_1^{2g}$		$u_{2g}^{2g} + i v_{2g}^{2g}$

where the u, v are the real and imaginary parts of the periods, and are, therefore, real. Then the column vectors of the above matrix are our periods $A_1, \dots, A_{2g}$. We wish to find g complex numbers $z_k = s_k + iy_k$ ($k = 1, \dots, g$) and a dfk $\varphi = z_1 \varphi_1 + \dots + z_g \varphi_g$ having the required periods. This amounts to solving, say for $i = 1$, the system of equations

$$\begin{aligned} x_1 u_1^i - y_1 v_1^i + \dots + x_g u_1^i - y_g v_1^i &= 1 \\ x_1 u_2^i - y_1 v_2^i + \dots + x_g u_2^i - y_g v_2^i &= 0 \\ &\dots \end{aligned}$$

This is solvable if the row vectors of the coefficient matrix are linearly independent over the reals. This is indeed the case, because a linear relation is immediately seen to imply a linear relation between $A_1, \dots, A_{2g}$ over the reals.

LEMMA 5. A dfk cannot have all its periods pure imaginary.

Proof. The preceding system of linear equations cannot have a solution when made homogenous.

THEOREM 11. Given a divisor $a = \sum n_P P$ of degree 0, there exists a unique dfk ω_a such that

1. $\text{res}_P \omega_a = n_P$ for all P .
2. The periods of ω_a are all pure imaginary.

Proof. Let ω be a dfk with $\text{res}_P \omega = n_P$. It suffices to find a dfk φ having the same real parts for the canonical periods (integrals around a_i). This is immediate from Lemma 4. The uniqueness is clear, because the difference between two differentials satisfying the conditions of the theorem would be a dfk contradicting Lemma 5.

THEOREM 12. If σ is a cycle such that $\int_\sigma \varphi = 0$ for all dfk then $\sigma \sim 0$.

Proof. Let $\sigma \sim n_1 a_1 + \dots + n_{2g} a_{2g}$. Find a dfk φ having periods

$$\operatorname{Re} \int_{a_1} \varphi = 1 \quad \text{and} \quad \operatorname{Re} \int_{a_j} \varphi = 0 \quad j \neq 1$$

Then $0 = \int_{\sigma} \varphi = n_1 + \text{pure imaginary}$, and so $n_1 = 0$.
Similarly, $n_j = 0$ all j .

THEOREM 13. The pairing

$$(\sigma, \mathfrak{a}) \rightarrow \langle \sigma, \mathfrak{a} \rangle = \exp \int_{\sigma} \omega_{\mathfrak{a}}$$

induces a bilinear pairing between the first homology group $H_1(R)$ and the group of divisor classes of degree 0. Its kernels on both sides are 0, and the pairing induces the Pontrajagin duality between the discrete group $H_1(R)$ and the compact torus.

Proof. Let D_0 be the group of divisors of degree 0. If $\mathfrak{a} \in D_0$ and $\mathfrak{a} = (z)$ is the divisor of a function, then we may take $\omega_{\mathfrak{a}} = dz/z$. All the periods of dz/z are pure imaginary, of the form $2\pi\sqrt{-1}m$ with some integer m . Hence $\langle \sigma, \mathfrak{a} \rangle = 1$ for such divisors, and all cycles σ having no point in common with $\mathfrak{a}$. Furthermore, if $\sigma \sim 0$, then

$$\begin{aligned} \int_{\sigma} \omega_{\mathfrak{a}} &= 2\pi\sqrt{-1} \sum \operatorname{res}_P \omega_{\mathfrak{a}} \\ &= 2\pi\sqrt{-1} (\text{integer}) \end{aligned}$$

so $\langle \sigma, \mathfrak{a} \rangle = 1$.

Our pairing is therefore well defined as a bilinear map of $H_1(R) \times J$ into the circle.

If $\mathfrak{a}$ is orthogonal to $H_1(R)$, then by definition, all periods of $\omega_{\mathfrak{a}}$ are of the type $2\pi\sqrt{-1}m$ for some integer m . We can then define a function z by letting

$$z(P) = \exp \int_O^P \omega_{\mathfrak{a}}$$

This function is meromorphic, and its divisor is $\mathfrak{a}$. Hence

the kernel on the right consists precisely of the group of divisors of functions.

To show that the kernel on the left consists precisely of these cycles σ which are ~ 0 , let us take an arbitrary cycle $\sigma \sim \sum n_i a_i$. We must show that all $n_i = 0$. Let $M_1, \dots, M_g$ be non-special points as in the proof of Jacobi's Theorem, and let us assume without loss of generality that these points do not lie on the cycles a_i . We consider divisors of degree 0 of type

$$a = P_1 + \dots + P_g - M_1 - \dots - M_g$$

where $P_1, \dots, P_g$ lie in small neighborhoods $V_1, \dots, V_g$ of $M_1, \dots, M_g$ respectively. Thus

$$(P_1, \dots, P_g) \in V_1 \times \dots \times V_g \subset \mathbb{C}^g$$

We put

$$w_i(a) = \int_{a_i} \omega_a$$

so that

$$\begin{aligned} w_1(a)A_1 + \dots + w_{2g}(a)A_{2g} &= 2\pi\sqrt{-1} \sum_{\nu=1}^g \int_{M_\nu}^{P_\nu} F \\ &= 2\pi\sqrt{-1} \sum (F(P_\nu) - F(M_\nu)) \end{aligned}$$

As a ranges over our divisors as prescribed above, we see from the Jacobi theorem that the vectors on the left hand side of our equation have real dimension $2g$. If $\langle \sigma, a \rangle = 0$ then

$$0 = \langle \sigma, a \rangle = \sum n_i \int_{a_i} \omega_a = \sum n_i w_i(a)$$

If not all $n_i = 0$, then the vectors $(w_1(\mathfrak{a}), \dots, w_{2g}(\mathfrak{a}))$ lie in a hyperplane, contradiction.

We shall leave the statement concerning the continuity of the pairing as an exercise for the reader.



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